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تئوری ورق و پوسته　　　تکلیف دوم ۲

جواب سوال ۱) با توجه به شرایط مرزی از روش حل نیمه تحلیلی لووی استفاده می‌شود.

$$w(x,y)=\sum_{m=1}^{\infty} Y_m(y)\,\sin\frac{m\pi x}{a}$$

$$p(x,y)=\sum_{m=1}^{\infty} P_m(y)\,\sin\frac{m\pi x}{a} \longrightarrow P_m(y)=\frac{2}{a}\int_0^a p(x,y)\sin\frac{m\pi x}{a}\,dx$$

$$\Rightarrow\quad P_m(y)=\frac{2}{a}\int_0^a P_0\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{m\pi x}{a}\right)dx$$

$$m=1 \longrightarrow \quad P_m(y)=\frac{2P_0}{a}\left[\frac{x}{2}-\frac{a}{4\pi}\sin\left(\frac{2\pi x}{a}\right)\right]_0^a$$

$$\longrightarrow \quad P_m(y)=\frac{2P_0}{a}\left[\frac{a}{2}-0-(0-0)\right]=P_0$$

$$\nabla^4 w=\frac{P}{D} \longrightarrow$$! مراحل حل در برگه دفترچه موجود است

$$\Rightarrow\quad Y_m(y)=f_m(y)+k_m(y)$$

$$\Rightarrow\quad w(x,y)=\sum_{m=1}^{\infty}\left[A_m\sinh\left(\frac{m\pi y}{a}\right)+B_m\cosh\left(\frac{m\pi y}{a}\right)+C_m y\sinh\left(\frac{m\pi y}{a}\right)+D_m y\cosh\left(\frac{m\pi y}{a}\right)+k_m\right]\times\sin\frac{m\pi x}{a}$$

$$m=1 \longrightarrow \quad w(x,y)=\left[A_m\sinh\left(\frac{\pi y}{a}\right)+B_m\cosh\left(\frac{\pi y}{a}\right)+C_m y\sinh\left(\frac{\pi y}{a}\right)+D_m y\cosh\left(\frac{\pi y}{a}\right)+k_m\right]\sin\left(\frac{\pi x}{a}\right)$$

$$* \; k_m=\frac{P_m a^4}{m^4\pi^4 D}=\frac{P_0 a^4}{m^4\pi^4 D}=\frac{P_0 a^4}{\pi^4 D}$$

شرایط مرزی

$$\begin{cases} y=-a/2 & w=0\,,\quad w_{,y}=0 \\ y=+a/2 & w=0\,,\quad w_{,y}=0 \end{cases}$$

$$\frac{dw}{dy}=w_{,y}=\left[A_m\frac{\pi}{a}\cosh\left(\frac{\pi y}{a}\right)+B_m\frac{\pi}{a}\sinh\left(\frac{\pi y}{a}\right)+C_m\left(\sinh\left(\frac{\pi y}{a}\right)+y\frac{\pi}{a}\cosh\left(\frac{\pi y}{a}\right)\right)+D_m\left(\cosh\left(\frac{\pi y}{a}\right)+y\frac{\pi}{a}\sinh\left(\frac{\pi y}{a}\right)\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

$$y=-a/2 \longrightarrow w=0 \longrightarrow -A_m\sinh\left(\frac{\pi}{2}\right)+B_m\cosh\left(\frac{\pi}{2}\right)+C_m\left(\frac{-a}{2}\right)\left(-\sinh\left(\frac{\pi}{2}\right)\right)+D_m\left(\frac{-a}{2}\right)\left(\cosh\left(\frac{\pi}{2}\right)\right)+\frac{P_0 a^4}{\pi^4 D}=0 \quad (I)$$

$$y=-a/2 \longrightarrow w_{,y}=0 \longrightarrow A_m\frac{\pi}{a}\cosh\left(\frac{\pi}{2}\right)-B_m\frac{\pi}{a}\sinh\left(\frac{\pi}{2}\right)+C_m\left(-\sinh\left(\frac{\pi}{2}\right)-\frac{a}{2}\times\frac{\pi}{a}\cosh\left(\frac{\pi}{2}\right)\right)+D_m\left(\cosh\left(\frac{\pi}{2}\right)-\frac{a}{2}\times\frac{\pi}{a}\sinh\left(\frac{\pi}{2}\right)\right)=0 \quad (II)$$

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$$y = a/2 \rightarrow w = 0 \longrightarrow A_m \sinh\left(\frac{\pi}{2}\right) + B_m \cosh\left(\frac{\pi}{2}\right) + C_m \frac{a}{2} \sinh\left(\frac{\pi}{2}\right) + D_m \frac{a}{2} \cosh\left(\frac{\pi}{2}\right) +$$

$$+ \frac{P_0 a^4}{\pi^4 D} = 0 \quad \text{(III)}$$

$$y = a/2 \rightarrow w_{,y} = 0 \longrightarrow A_m \frac{\pi}{a} \cosh\left(\frac{\pi}{2}\right) + B_m \frac{\pi}{a} \sinh\left(\frac{\pi}{2}\right) + C_m \left(\sinh\left(\frac{\pi}{2}\right) + \frac{a}{2} \times \frac{\pi}{a} \cosh\left(\frac{\pi}{2}\right)\right) +$$

$$D_m \left(\cosh\left(\frac{\pi}{2}\right) + \frac{a}{2} \alpha \frac{\pi}{a} \sinh\left(\frac{\pi}{2}\right)\right) = 0 \quad \text{(IV)}$$

تعريف
$$\begin{cases} \sinh\left(\frac{\pi}{2}\right) = \alpha \\ \cosh\left(\frac{\pi}{2}\right) = \beta \end{cases}$$

$$1: \text{(I)} - \text{(III)} \longrightarrow -2A_m\alpha - D_m a\beta = 0$$

$$2: \text{(II)} + \text{(IV)} \longrightarrow 2A_m \frac{\pi}{a}\beta + 2D_m\beta = 0$$

$$3: \text{(I)} + \text{(III)} \longrightarrow 2B_m\beta + C_m a\alpha = \frac{-2P_0 a^4}{\pi^4 D}$$

$$4: \text{(II)} - \text{(IV)} \longrightarrow -2B_m \frac{\pi}{a}\alpha - 2C_m\left(\alpha + \frac{\pi}{2}\beta\right) - D_m \pi\alpha = 0$$

$$2 \Rightarrow \quad A_m = D_m = 0$$

$$3 \Rightarrow \quad 2B_m\beta + C_m a\alpha = \frac{-2P_0 a^4}{\pi^4 D}$$

$$4 \Rightarrow \quad -2B_m \frac{\pi}{a}\alpha - 2C_m\left(\alpha + \frac{\pi}{2}\beta\right) = 0 \quad \Rightarrow \quad B_m = C_m \frac{a}{\pi}\left(\alpha + \frac{\pi}{2}\beta\right)$$

$$\rightsquigarrow \left(2\beta \frac{a}{\pi}\left(\alpha + \frac{\pi}{2}\beta\right) + a\alpha\right) C_m = \frac{-2P_0 a^4}{\pi^4 D}$$

$$\Rightarrow \quad C_m = \frac{-2P_0 a^3}{\pi^4 D\left[\frac{2\beta}{\pi}\left(\alpha + \frac{\pi}{2}\beta\right) + \alpha\right]}$$

$$\Rightarrow \quad B_m = \frac{-2P_0 a^4\left(\alpha + \frac{\pi}{2}\beta\right)}{\pi^5 D\left[\frac{2\beta}{\pi}\left(\alpha + \frac{\pi}{2}\beta\right) + \alpha\right]}$$

$$\Rightarrow \quad w(x,y) = \left[B_m \cosh\left(\frac{\pi y}{a}\right) + C_m y \sinh\left(\frac{\pi y}{a}\right) + \frac{P_0 a^4}{\pi^4 D}\right] \sin\left(\frac{\pi x}{a}\right) \quad \checkmark \text{ (ب)}$$

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ماکزیمم خیز ورق با توجه به تقارن حول وسط ورق رخ می دهد.

$$y = 0 \ , \ x = \frac{a}{2}$$

$$\Rightarrow w_{max} = \left[B_m \times 1 - C_m \times 0 \times 0 + \frac{P_0 a^4}{\pi^4 D} \right] \times \sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow w_{max} = \left[B_m + \frac{P_0 a^4}{\pi^4 D} \right] \times 1 = \frac{P_0 a^4}{\pi^4 D} \left[1 - \frac{2\left(\alpha + \frac{\pi}{2}\beta\right)}{\pi\left[\frac{2\beta}{\pi}\left(\alpha + \frac{\pi}{2}\beta\right) + \alpha\right]} \right] \quad \checkmark \text{ (ب)}$$

$$M_x = -D\left(w_{,xx} + \nu w_{,yy}\right)$$

$$M_y = -D\left(w_{,yy} + \nu w_{,xx}\right)$$

$$\rightarrow w_{,x} = \frac{\pi}{a}\cos\left(\frac{\pi x}{a}\right)\left[B_m \cosh\left(\frac{\pi y}{a}\right) + C_m y \sinh\left(\frac{\pi y}{a}\right) + \frac{P_0 a^4}{\pi^4 D} \right]$$

$$\rightarrow w_{,xx} = -\left(\frac{\pi}{a}\right)^2 \sin\left(\frac{\pi x}{a}\right)\left[B_m \cosh\left(\frac{\pi y}{a}\right) + C_m y \sinh\left(\frac{\pi y}{a}\right) + \frac{P_0 a^4}{\pi^4 D} \right]$$

$$\rightarrow w_{,y} = \sin\left(\frac{\pi x}{a}\right)\left[\frac{\pi}{a} B_m \sinh\left(\frac{\pi y}{a}\right) + C_m\left(\sinh\left(\frac{\pi y}{a}\right) + \underbrace{y}_{\pi/a} \cosh\left(\frac{\pi y}{a}\right) \right) \right]$$

$$\rightarrow w_{,yy} = \sin\left(\frac{\pi x}{a}\right)\left[\left(\frac{\pi}{a}\right)^2 B_m \cosh\left(\frac{\pi y}{a}\right) + C_m\left(\frac{\pi}{a}\cosh\left(\frac{\pi y}{a}\right) + \frac{\pi}{a}\cos\left(\frac{\pi y}{a}\right) + \left(\frac{\pi}{a}\right)^2 y \sin\left(\frac{\pi y}{a}\right) \right) \right]$$

$$\nu = 0.3 \ , \ y = 0 \ , \ x = \frac{a}{2}$$

$$\rightarrow w_{,x} = 0$$

$$\rightarrow w_{,xx} = -\left(\frac{\pi}{a}\right)^2 \left[B_m \times 1 + 0 + \frac{P_0 a^4}{\pi^4 D} \right] = -\left(\frac{\pi}{a}\right)^2 \left[B_m + \frac{P_0 a^4}{\pi^4 D} \right]$$

$$\rightarrow w_{,y} = 0$$

$$\rightarrow w_{,yy} = 1 \times \left[\left(\frac{\pi}{a}\right)^2 B_m \times 1 + C_m\left(\frac{\pi}{a} \times 1 + \frac{\pi}{a} \times 1 + 0 \right) \right] = \left(\frac{\pi}{a}\right)^2 B_m + 2 C_m \frac{\pi}{a}$$

$$M_x = -D\left[-\left(\frac{\pi}{a}\right)^2 \left(B_m + \frac{P_0 a^4}{\pi^4 D} \right) + 0.3\left(\left(\frac{\pi}{a}\right)^2 B_m + 2 C_m \frac{\pi}{a} \right) \right]$$

$$M_y = -D\left[\left(\frac{\pi}{a}\right)^2 B_m + 2 C_m \frac{\pi}{a} \bullet -0.3 \left(\frac{\pi}{a}\right)^2 \left(B_m + \frac{P_0 a^4}{\pi^4 D} \right) \right]$$

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جواب سوال ۲)

تکیه‌گاه ساده $\rightarrow$ مرزی $\rightarrow$ $w = 0 \ , \ w_{,xx} = w_{,yy} = 0$

تکیه‌گاه گیردار $\rightarrow$ مرزی $\rightarrow$ $w = 0 \ , \ w_{,x} = w_{,y} = 0$

$$w(x,y) = \frac{P_0 x}{24D}\left[\frac{\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)^2}{\frac{5}{a^4}+\frac{1}{b^4}+\frac{2}{a^2b^2}}\right]$$

معادله خط مرزی مرز ورق همان معادله خط بیضی است:

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$$

در نتیجه خیز ورق با توجه به فرم پیشنهادی آن در مرز ورق صفر است و شرط در مورد حالت ساده و گیردار صدق می‌کند!

$$\rightarrow w_{,x} = \frac{P_0}{24D}\Bigg[\underbrace{\frac{1}{\frac{5}{a^4}+\frac{1}{b^4}+\frac{2}{a^2b^2}}}_{\alpha}\Bigg]\times\left[\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)^2 - x\times 2\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)\times 2\left(\frac{x}{a}\right)\times\frac{1}{a}\right]$$

$$\rightarrow w_{,x} = \frac{P_0}{24D\alpha}\left[\underbrace{\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)^2}_{\text{معادله خط بیضی}} - 4\frac{x^2}{a^2}\times\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)\right]$$

$$\rightarrow w_{,xx} = \frac{P_0}{24D\alpha}\left[-2\times\frac{2}{a}\times\left(\frac{x}{a}\right)\underbrace{\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)}_{\text{معادله خط بیضی}} - \frac{4}{a^2}\left[2x\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right) - \underbrace{x^2\times\frac{2}{a}\left(\frac{x}{a}\right)}_{\text{یک ترم اضافه}}\right]\right]$$

با توجه به وجود یک ترم اضافه در مشتق دوم خیز $(w_{,xx})$ عملاً شرایط تکیه‌گاه گیردار صدق نمی‌کند!

$$\rightarrow w_{,y} = \frac{P_0 x}{24D\alpha}\left[-2\times\frac{2}{b}\left(\frac{y}{b}\right)\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)\right]$$

$$\rightarrow w_{,y} = \frac{-P_0 x}{6D\alpha b^2}\left[y\underbrace{\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)}_{\text{معادله خط بیضی}}\right]$$

$$\rightarrow w_{,yy} = \frac{-P_0 x}{6D\alpha b^2}\left[\underbrace{\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)}_{\text{معادله خط بیضی}} - \underbrace{y\times\frac{2}{b}\left(\frac{y}{b}\right)}_{\text{یک ترم اضافه}}\right]$$

در این مشتق دوم خیز نسبت به y هم چون شرط قبل نقض می‌شود و در مجموع می‌توان گفت که معادله خیز ورق پیشنهادی برای تکیه‌گاه ساده صادق نیست
اما برای تکیه‌گاه گیردار صدق نمی‌کند!

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$$\nabla^4 w - \frac{P}{D} \rightarrow \quad w_{,nnnn} + 2\,w_{,nnyy} + w_{,yyyy} = \frac{P_0 x}{D}$$

$$\rightarrow w_{,nnn} = \frac{P_0}{24 D \alpha}\left[\frac{-4}{a}\left(\frac{1}{a}\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)-\frac{x}{a}\times 2\frac{1}{a}\left(\frac{x}{a}\right)\right) -\right.$$

$$\left. \hookrightarrow \frac{-4}{a^2}\left[2\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)-2x\times 2\frac{1}{a}\left(\frac{x}{a}\right)-\frac{6}{a^2}x^2\right]\right]$$

$$\rightarrow w_{,nnnn} = \frac{P_0}{24 D \alpha}\left\{\frac{-4}{a}\times\left(\frac{1}{a}\times\frac{-2}{a}\left(\frac{x}{a}\right)-\frac{4}{a^3}x\right) -\right.$$

$$\left. \hookrightarrow \frac{-4}{a^2}\left[-2\times 2\frac{1}{a}\left(\frac{x}{a}\right)-\frac{8}{a^2}x-\frac{12}{a^2}x\right]\right\}$$

$$\rightarrow w_{,nnnn} = \frac{P_0}{24 D \alpha}\left[\frac{-4}{a}\left(\frac{-2x}{a^3}-\frac{4x}{a^3}\right)-\frac{4}{a^2}\left[\frac{-4x}{a^2}-\frac{20x}{a^2}\right]\right]$$

$$\rightarrow w_{,nnnn} = \frac{5 P_0 x}{D \alpha a^4}$$

$$\rightarrow w_{,yyyy} = \frac{-P_0 x}{6 D \alpha b^2}\left[-2\times\frac{1}{b}\left(\frac{y}{b}\right)-\frac{4}{b^2}y\right] = \frac{P_0 xy}{D\alpha b^4}$$

$$\rightarrow w_{,yyyy} = \frac{P_0 x}{D \alpha b^4}$$

$$\rightarrow w_{,yyx} = \frac{-P_0}{6 D \alpha b^2}\left[\left(1-\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2\right)-\frac{2y^2}{b^2}-x\times 2\times\frac{1}{a}\left(\frac{x}{a}\right)\right]$$

$$\rightarrow w_{,yyxx} = \frac{-P_0}{6 D \alpha b^2}\left[-2\times\frac{1}{a}\left(\frac{x}{a}\right)-\frac{4x}{a^2}\right]$$

$$\rightarrow w_{,yynn} = \frac{P_0 x}{D \alpha b^2 a^2}$$

$$\nabla^4 w = \frac{P}{D} \Rightarrow \frac{5 P_0 x}{D\alpha a^4} + 2\frac{P_0 x}{D\alpha b^2 a^2} + \frac{P_0 x}{D\alpha b^4} = \frac{P_0 x}{D}$$

$$\Rightarrow \frac{5x}{\alpha a^4} + \frac{2x}{\alpha b^2 a^2} + \frac{x}{\alpha b^4} = x$$

$$\bigstar \quad \alpha = \frac{5}{a^4} + \frac{1}{b^4} + \frac{2}{a^2 b^2}$$

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$$\frac{5x}{a^4} + \frac{2x}{a^2b^2} + \frac{x}{b^4} = x\left(\frac{5}{a^4} + \frac{1}{b^4} + \frac{2}{a^2b^2}\right)$$

معادله پیشنهادی در معادله خیز ورق صدق می‌کند!

$$M_x = -D(w_{,xx} + \nu w_{,yy})$$

$$M_y = -D(w_{,yy} + \nu w_{,xx})$$

A نقطه $\longrightarrow$ $x = a,\ y = 0$

$$\rightarrow w_{,xx} = \frac{P_0}{24D\alpha}\left[\frac{-4}{a} \times 0 - \frac{4}{a^2}\left[2a \times 0 - 2a\right]\right] = \frac{8P_0}{24D\alpha a}$$

$$\rightarrow w_{,yy} = \frac{-P_0 a}{6D\alpha b^2}\left[0 - 0 \times 0\right] = 0$$

$$\Rightarrow \begin{cases} M_x = -D \times \dfrac{8P_0}{24D\alpha a} = \dfrac{-P_0}{3\alpha a} \\ M_y = -D\nu \times \dfrac{8P_0}{24D\alpha a} = \dfrac{-P_0\nu}{3\alpha a} \end{cases}$$

$$a = 2b \longrightarrow \alpha = \frac{5}{a^4} + \frac{1}{b^4} + \frac{2}{a^2b^2} = \frac{5}{16b^4} + \frac{1}{b^4} + \frac{2}{8b^4} = \frac{25}{16b^4}$$

$$\Rightarrow \begin{cases} M_x = \dfrac{-P_0}{3 \times \frac{25}{16} \times \frac{1}{b^4} \times a} = \dfrac{-16P_0 b^4}{75a} \\ M_y = \dfrac{-P_0\nu}{3 \times \frac{25}{16} \times \frac{1}{b^4} \times a} = \dfrac{-16P_0 b^4\nu}{75a} \end{cases}$$